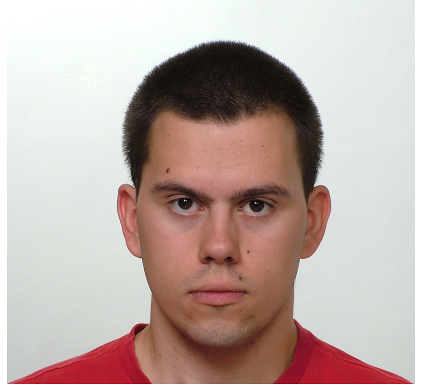
Shortcut Partition for Minor-Free Graphs: Steiner Point Removal and More


Hsien-Chih Chang


Jonathan Conroy


Hung Le


Lazar Milenković


Shay Solomon


Cuong Than

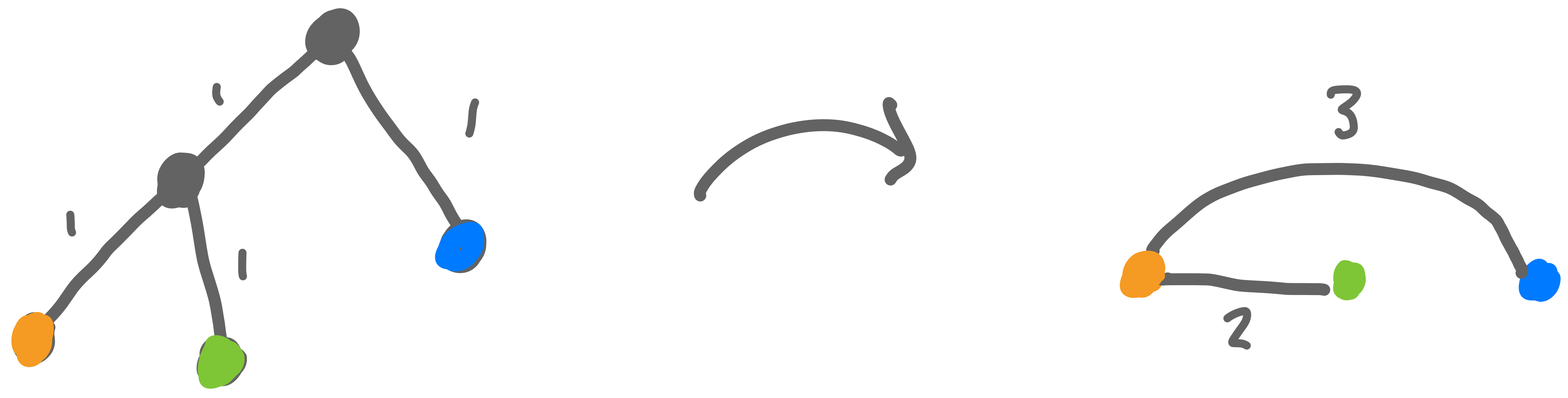
Steiner Point Removal

[Gup '01]

Input: Weighted tree G and set T of *terminal* vertices.

Output: Weighted tree G' with vertex set T, such that
$\forall x, y \in T$

$$d_G(x,y) \le d_{G'}(x,y) \le \alpha \cdot d_G(x,y)$$



Steiner Point Removal

[Gup '01]
[CXKR '06]

Input: Weighted tree G and set T of *terminal* vertices.

Output: Weighted ~~tree~~ minor G' of G with vertex set T, such that
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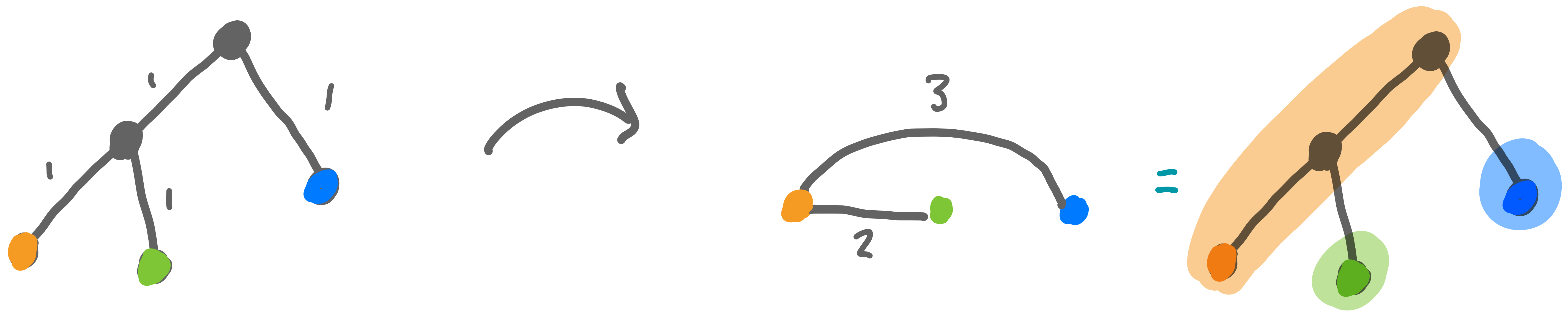

Steiner Point Removal

[Gup '01]
[CXKR '06]

Input: Weighted ~~tree~~ K_r-minor-free graph G and set T of *terminal* vertices.

Output: Weighted ~~tree~~ minor of G G' with vertex set T, such that

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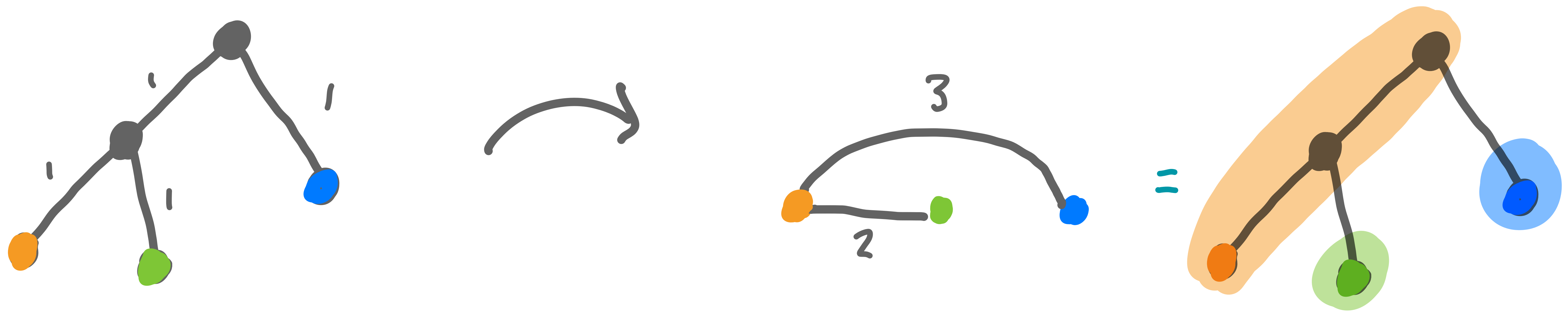

SPR: (Some) Prior Results

SPR Distortion

- Trees: 8 [Gup'01, CXKR'06]
- Outerplanar: $O(1)$ [BG'08]
- Series-Parallel: $O(1)$ [HL'22], using framework of [Fil'20]

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Question: Do planar — or, further, minor-free graphs — admit SPR solutions with $O(1)$ distortion?

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SPR Distortion

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Question: Do planar — or, further, minor-free graphs — admit SPR solutions with $O(1)$ distortion?

General graphs: $\omega(1)$ distortion [TC'24]

SPR: Our Results

Every planar graph admits an SPR solution with $O(1)$ distortion.

Every K_r-minor-free graph admits an SPR solution with $2^{O(r \log r)}$ distortion.

Outline

1. Steiner Point Removal

$\Uparrow$

2. Shortcut partition & the buffer property

$\Uparrow$

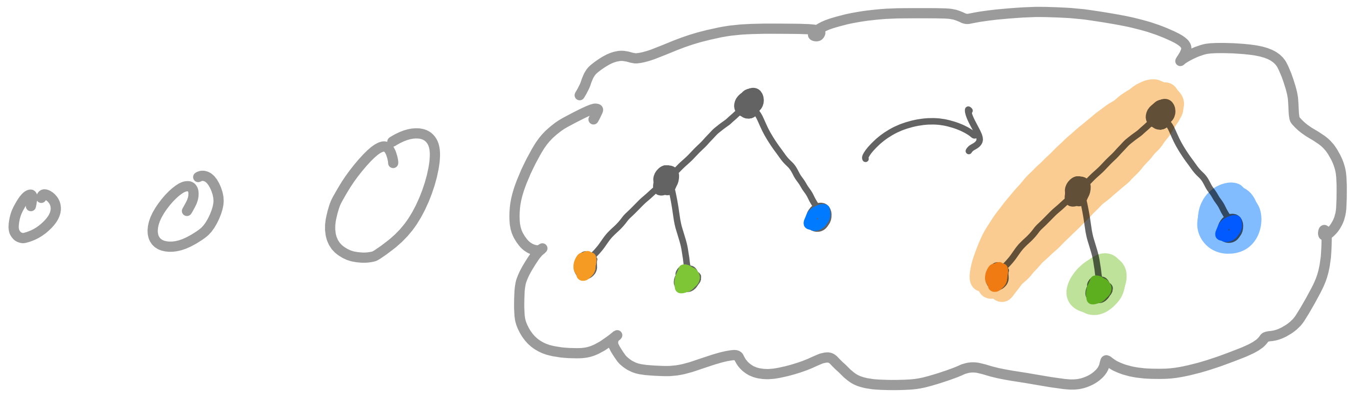
3. Cop-decomposition of [AGGNT'14]

4. Our deterministic modification

$\leftarrow$ Main technical contribution

Outline

1. Steiner Point Removal



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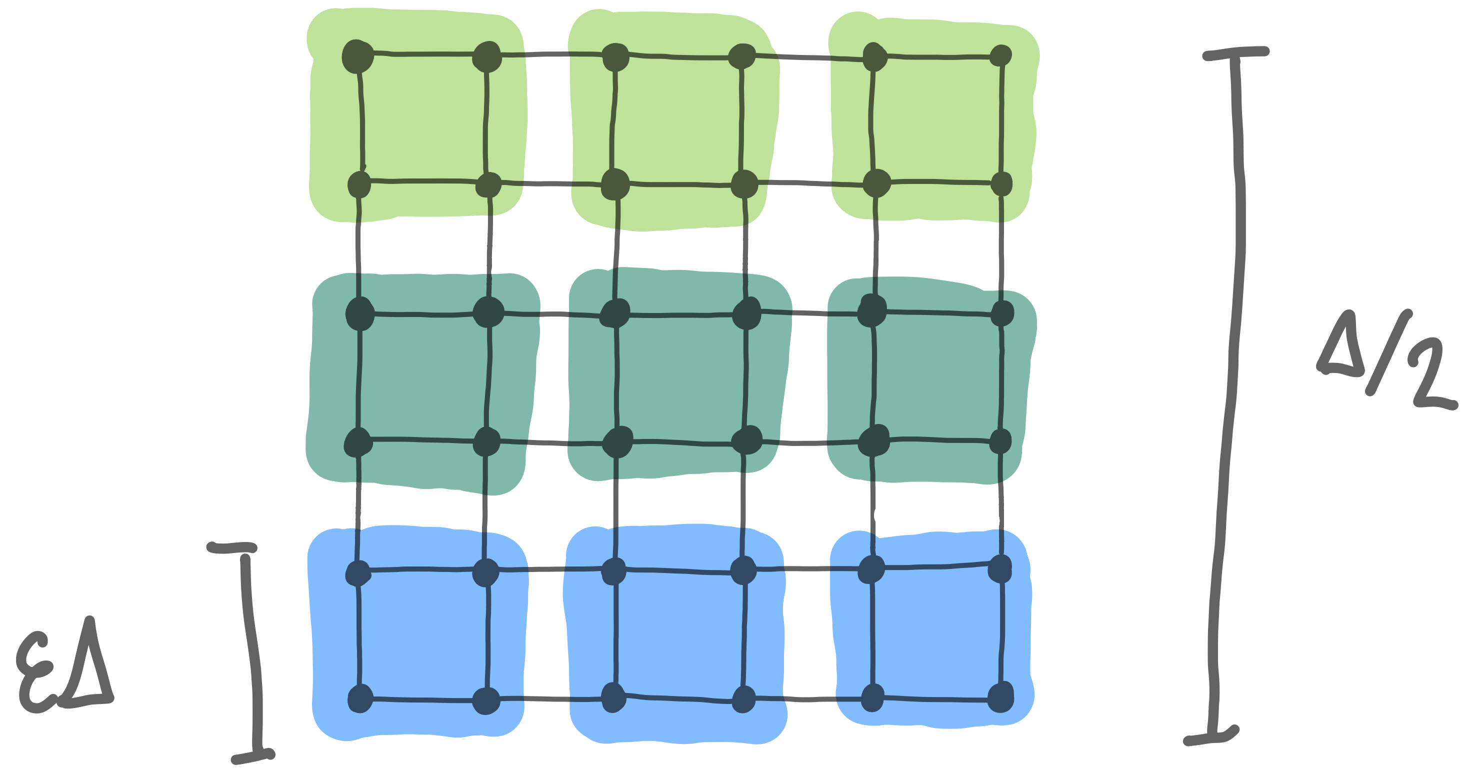
Scattering Partition

Given graph G with diameter Δ. Want to partition vertices into clusters of diameter $\varepsilon\Delta$, such that any shortest-path intersects only $O(1/\varepsilon)$ clusters



Scattering partition $\implies$ SPR with $O(1)$ distortion

[Filtser'20]

Shortcut ~~Scattering~~ Partition

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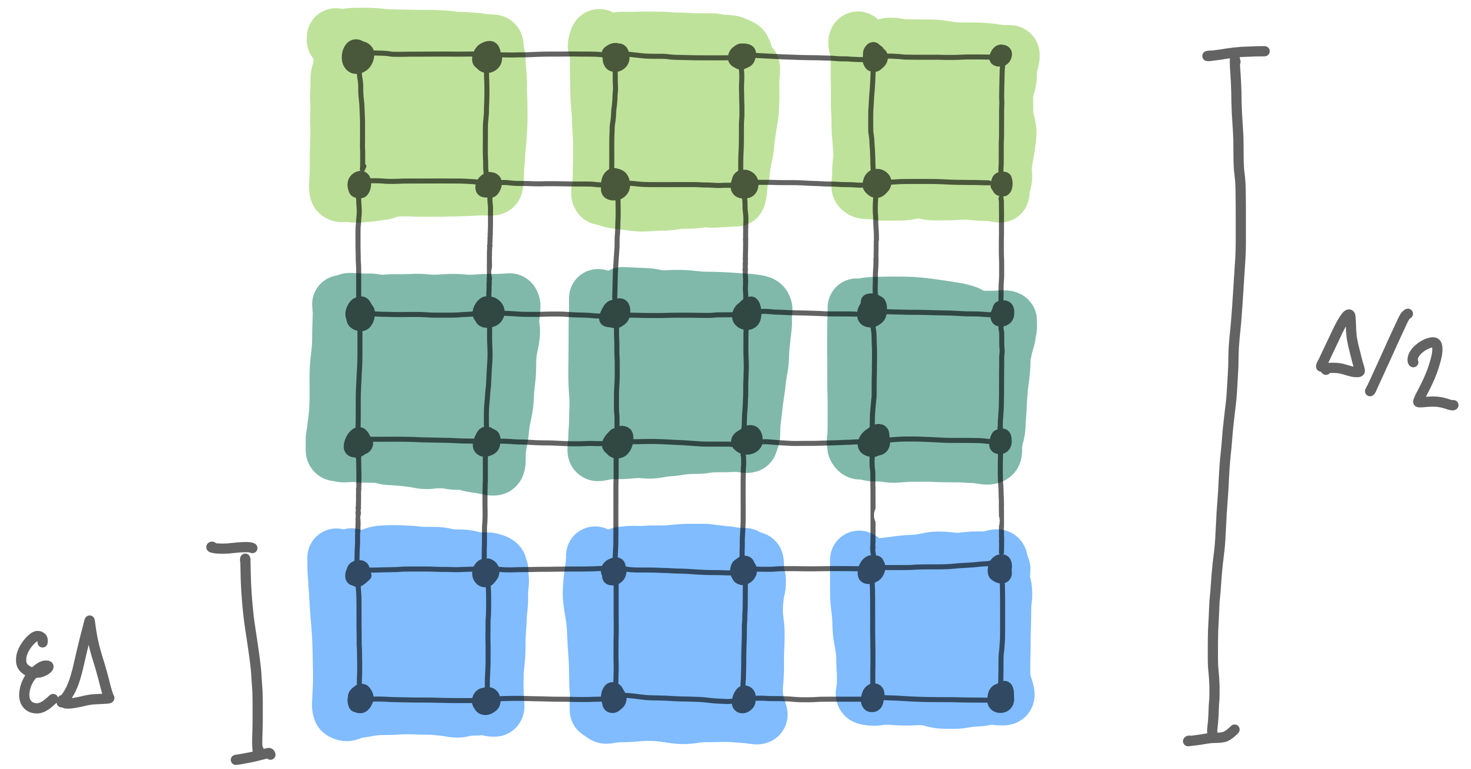

Scattering partition $\implies$ SPR with $O(1)$ distortion tree cover

[Filtser'20] [CCLMST'23] Planar graphs have shortcut partition → distance oracle

→ tree cover

→ ...

~~Scattering~~ Shortcut Partition

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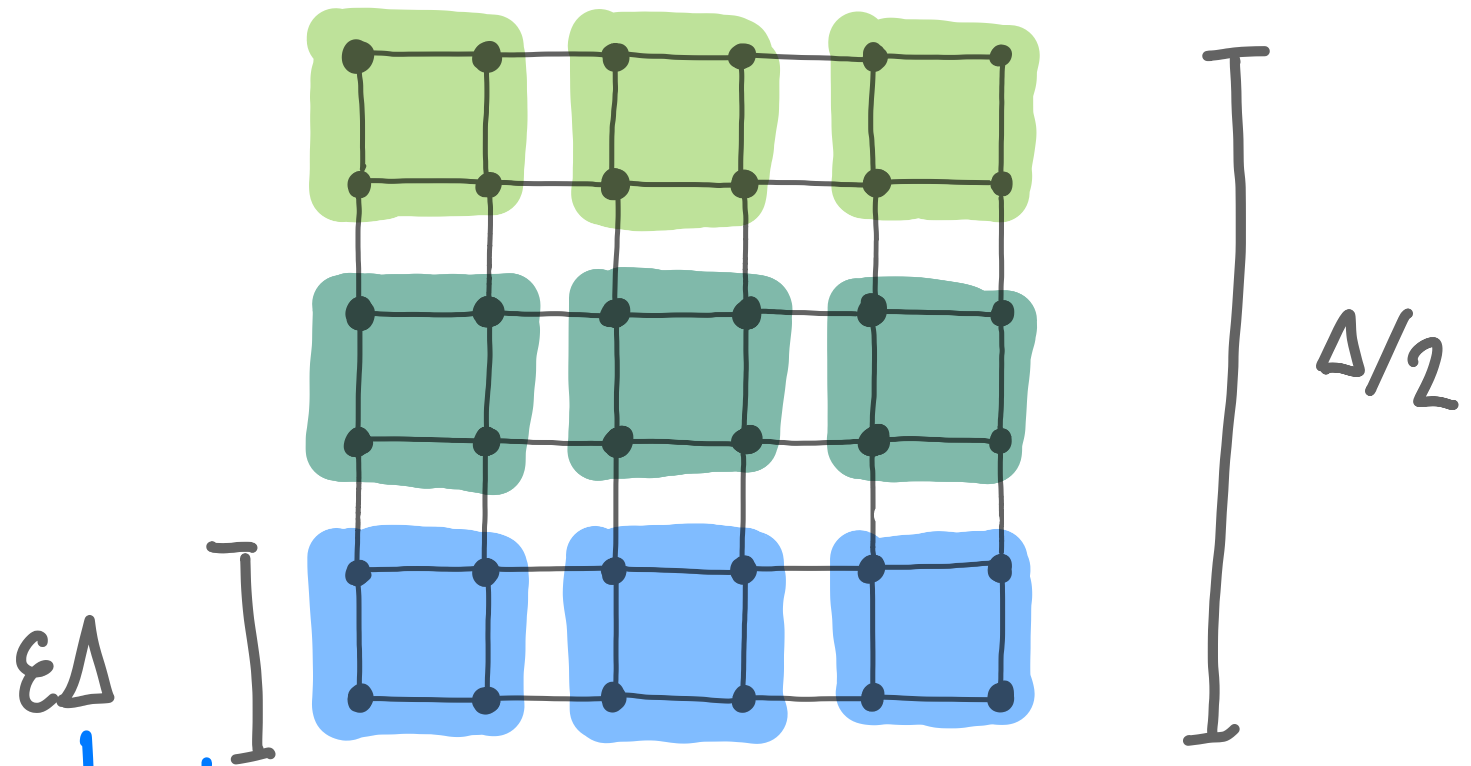
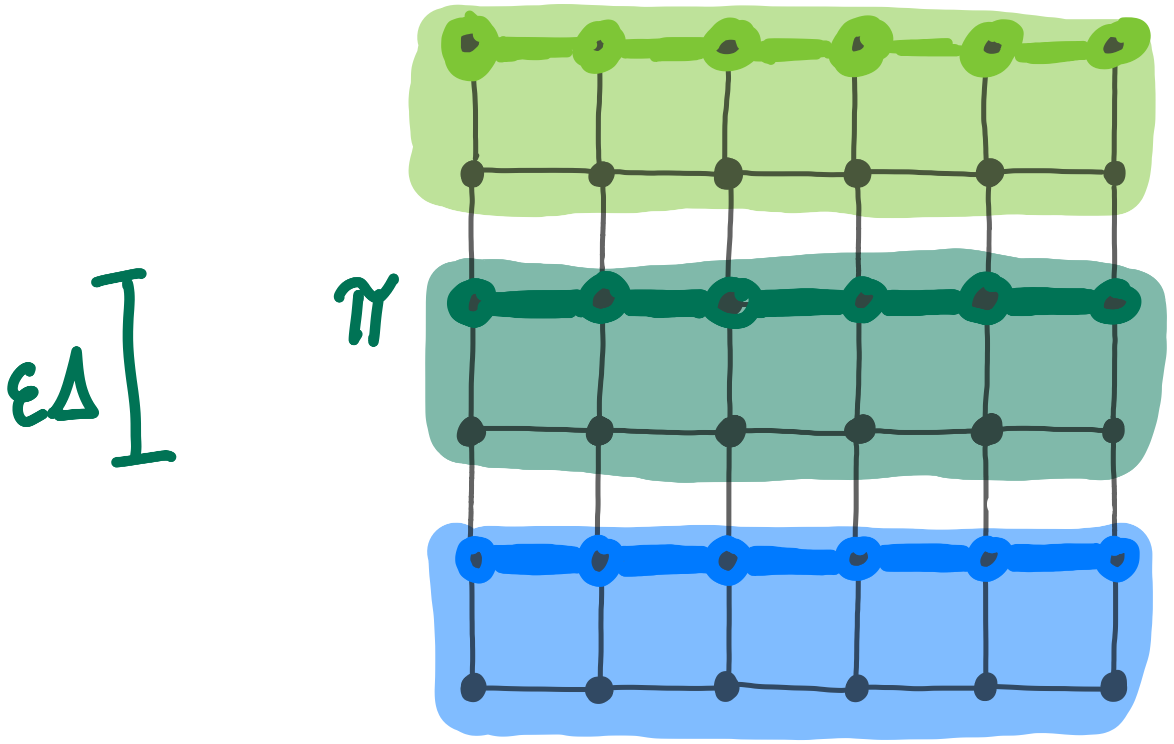


We show: ~~Scattering~~ Shortcut partition $\Longrightarrow$ SPR with $O(1)$ distortion tree cover

[Filtser'20] [CCLMST'23] Planar graphs have shortcut partition → distance oracle → ...

How to Get Shortcut Partition

[CCLMST'23] Partition graph into *expanded paths* (a shortest-path π and subset of points within distance $\varepsilon\Delta$ of π).



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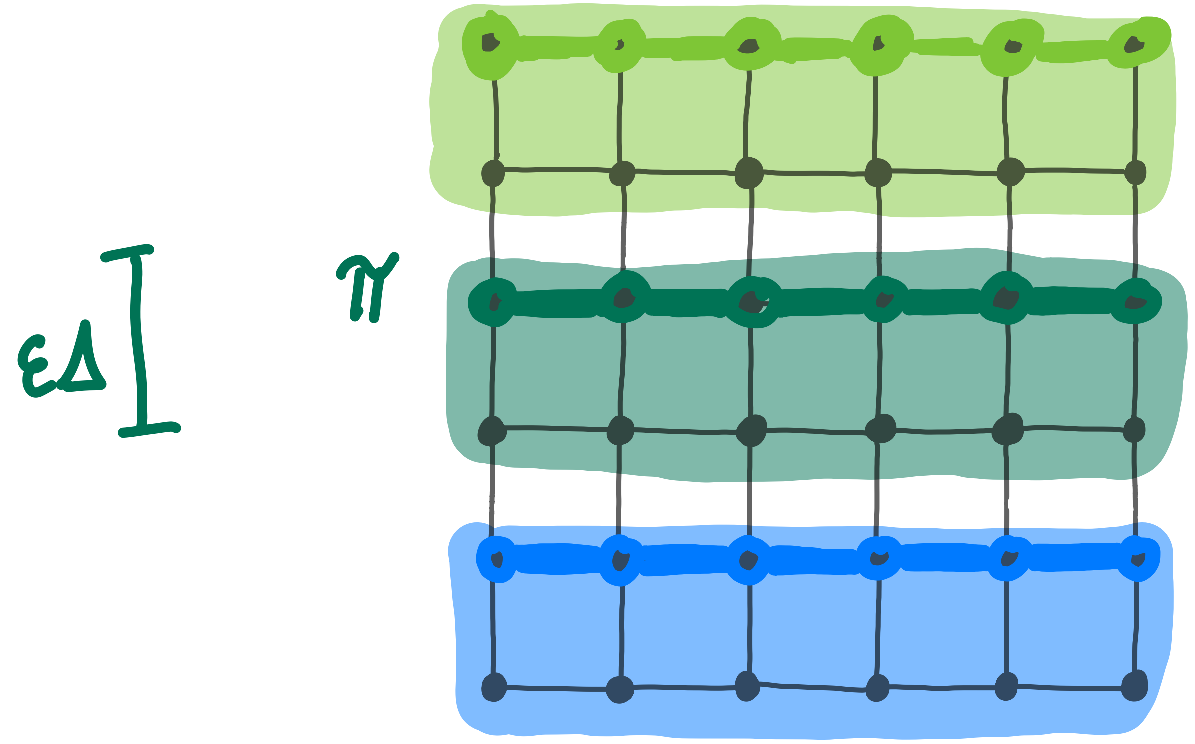

Buffer property (informal): If an expanded path X is "cut off" from a piece of the graph by another expanded path, then every path from X to that piece has length $\geq \varepsilon\Delta/O(1)$.

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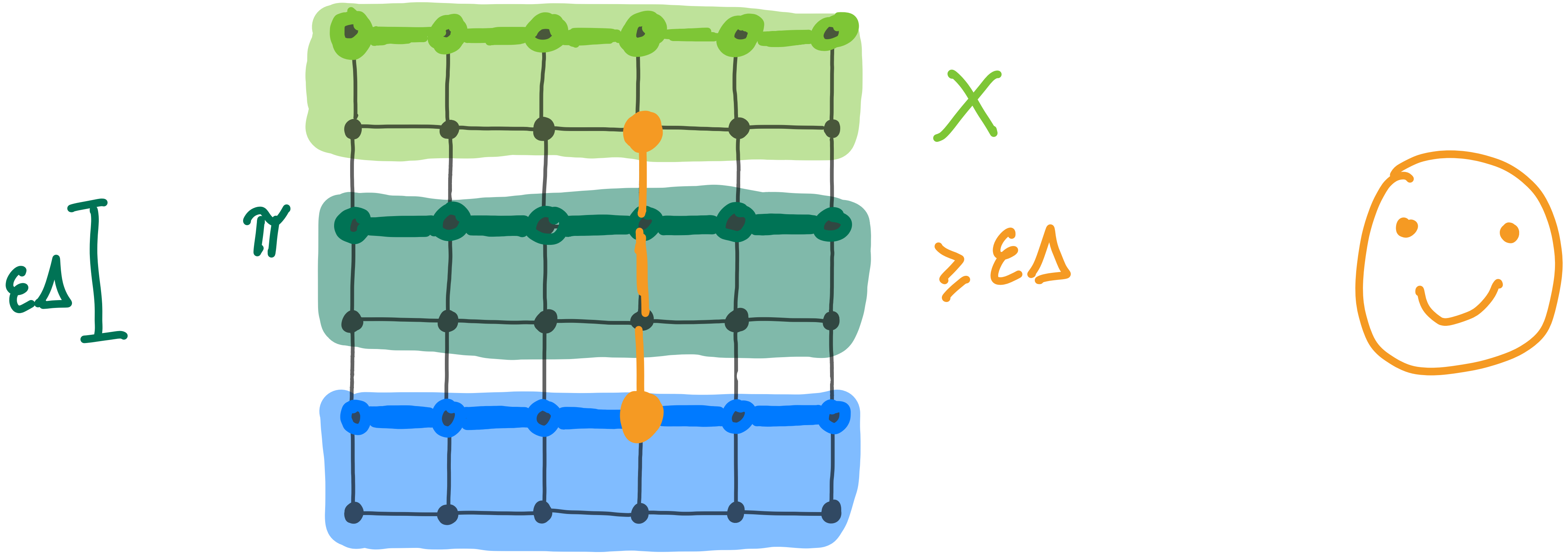

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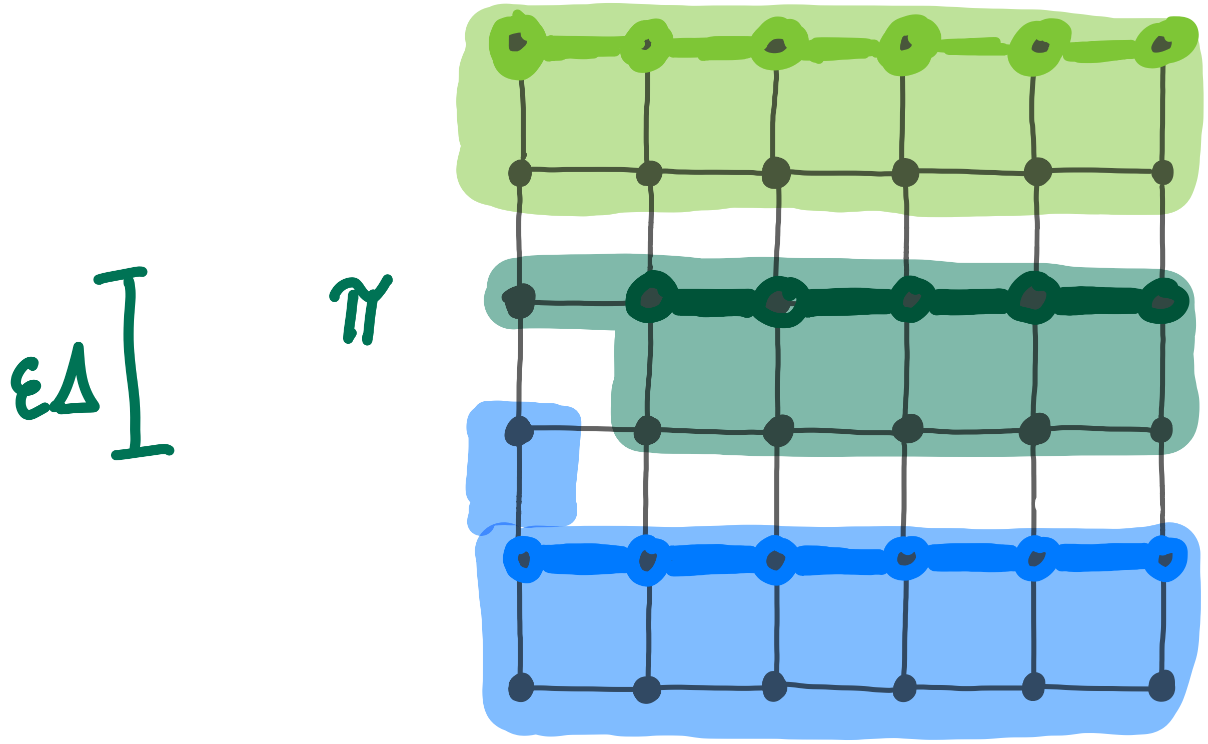

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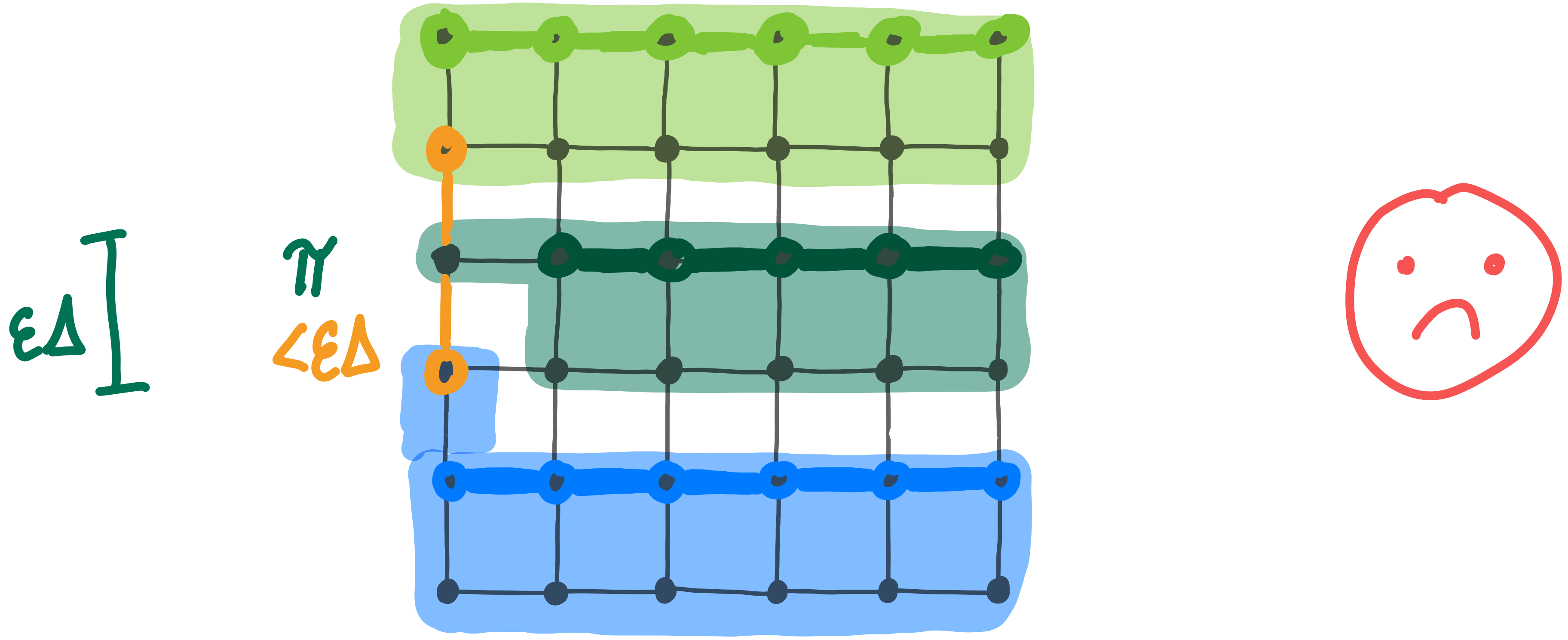

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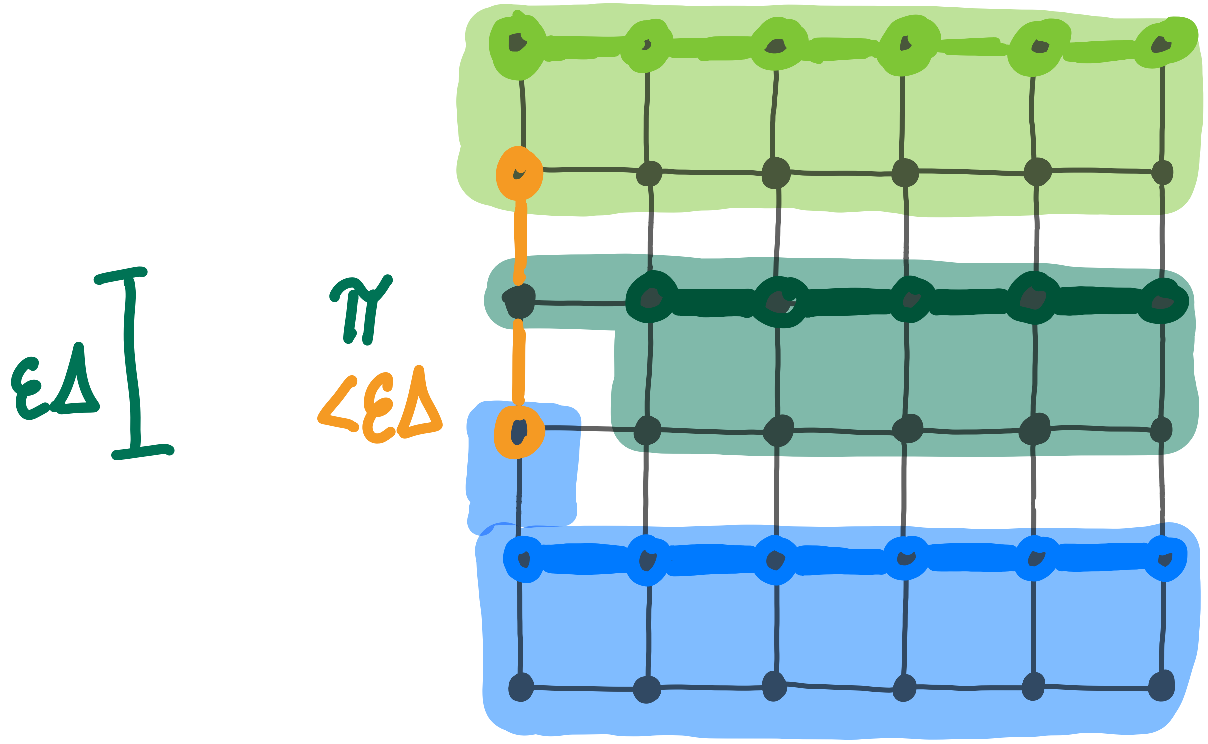

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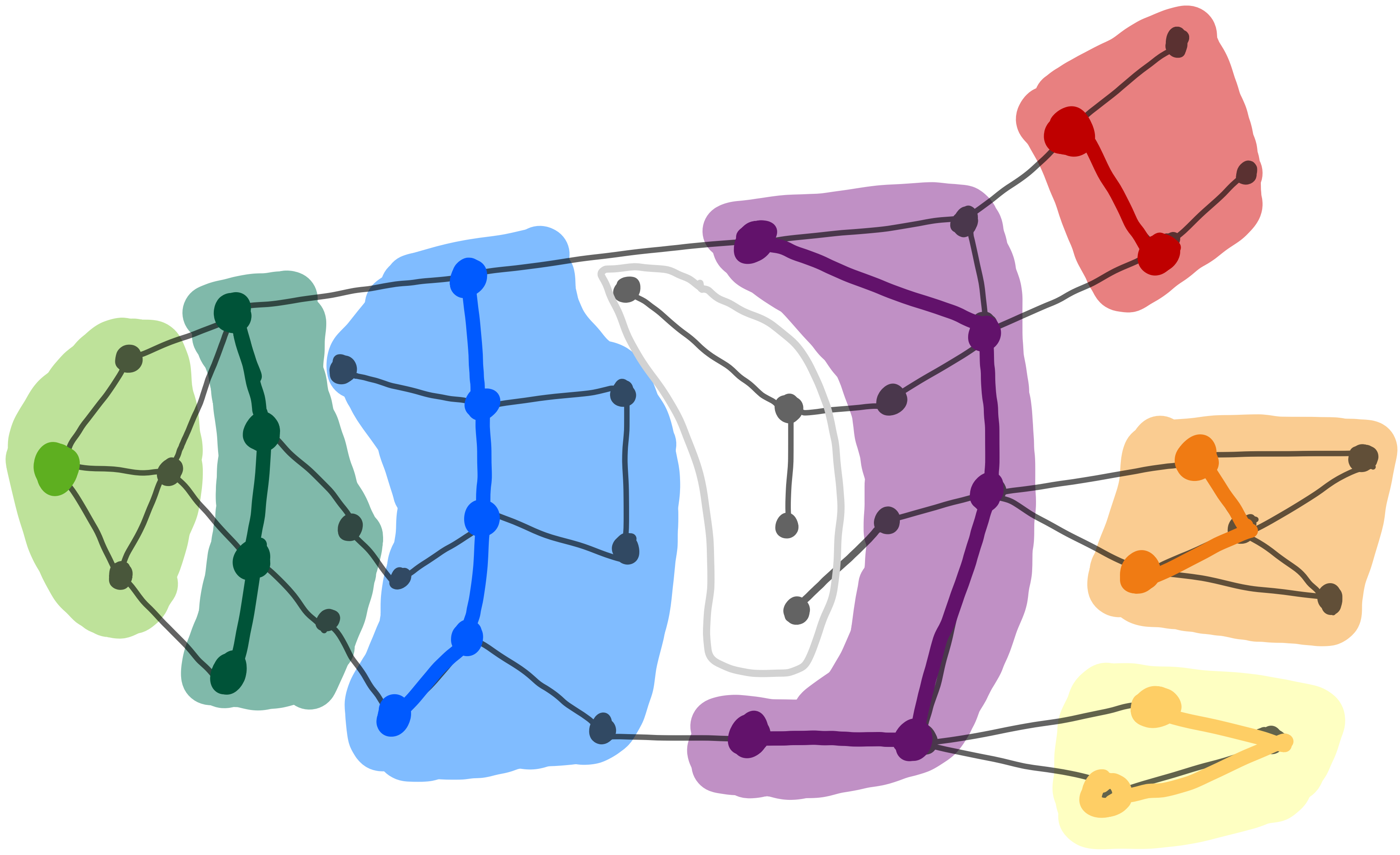
How to Get Shortcut Partition

Partition graph into *expanded paths* satisfying *buffer property*



[CCLMST'23] finds expanded paths by working along the *outer face*.

How to breach planarity barrier?

Outline

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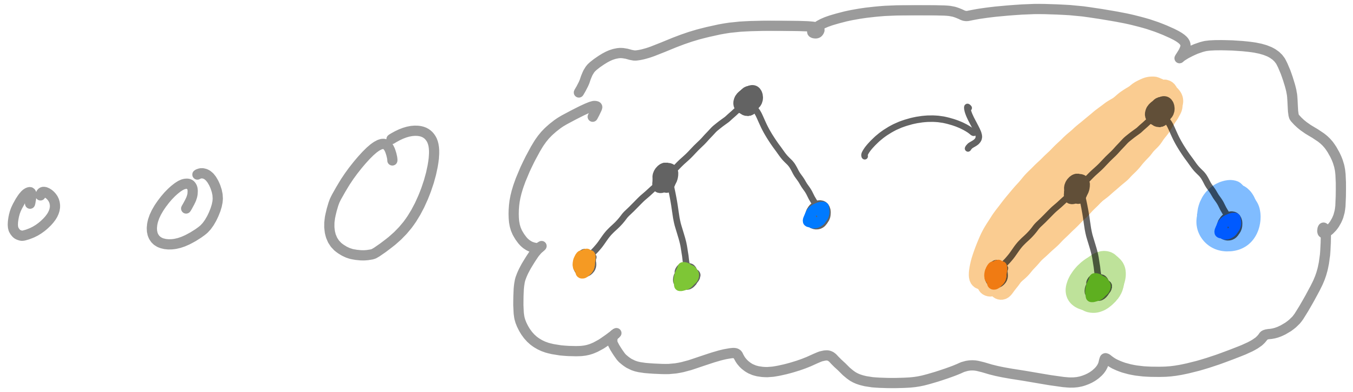
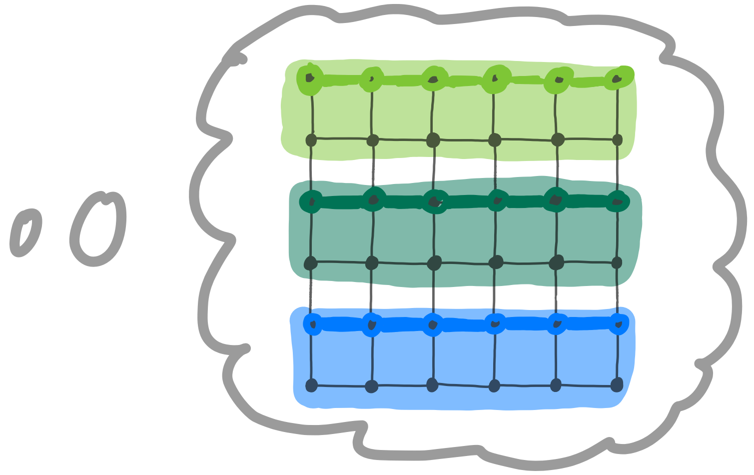


$\Uparrow$

2. Shortcut partition & the buffer property



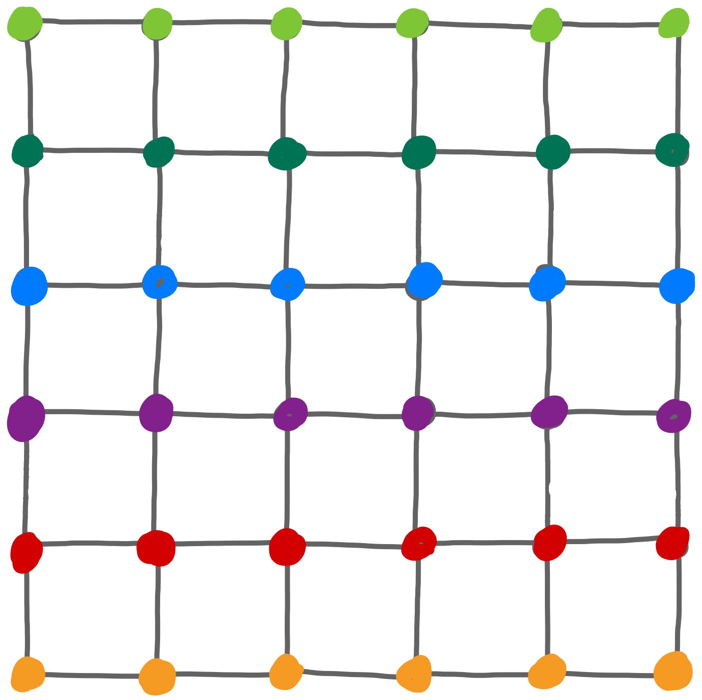
$\Uparrow$

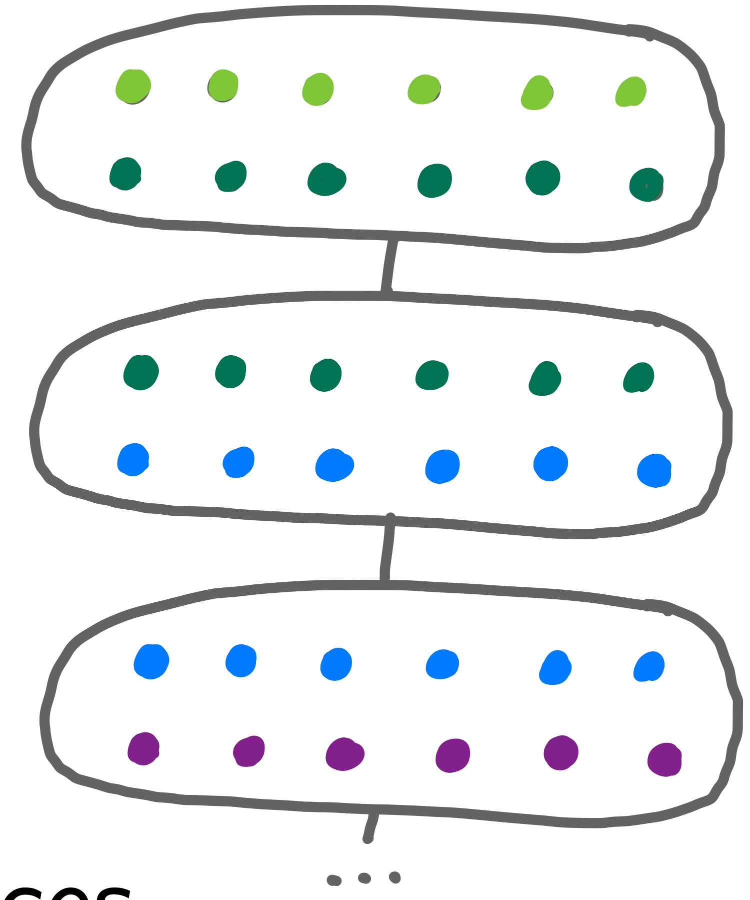
3. Cop-decomposition of [AGGNT'14]

4. Our deterministic modification

Cop-Decomposition

[Abraham-Gavoille-Gupta-Neiman-Talwar'14]





treewidth k: bags of $\leq k$ vertices

$O(\sqrt{n})$

Cop-Decomposition

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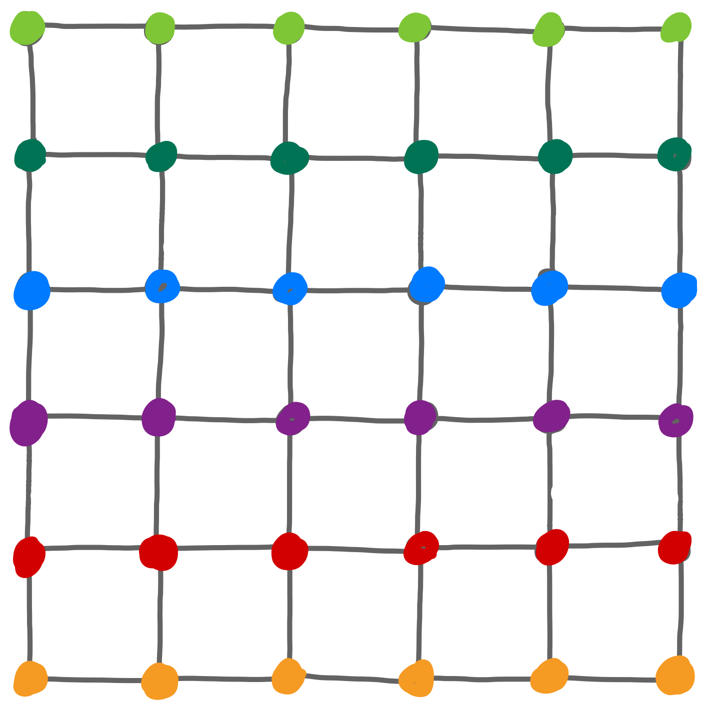
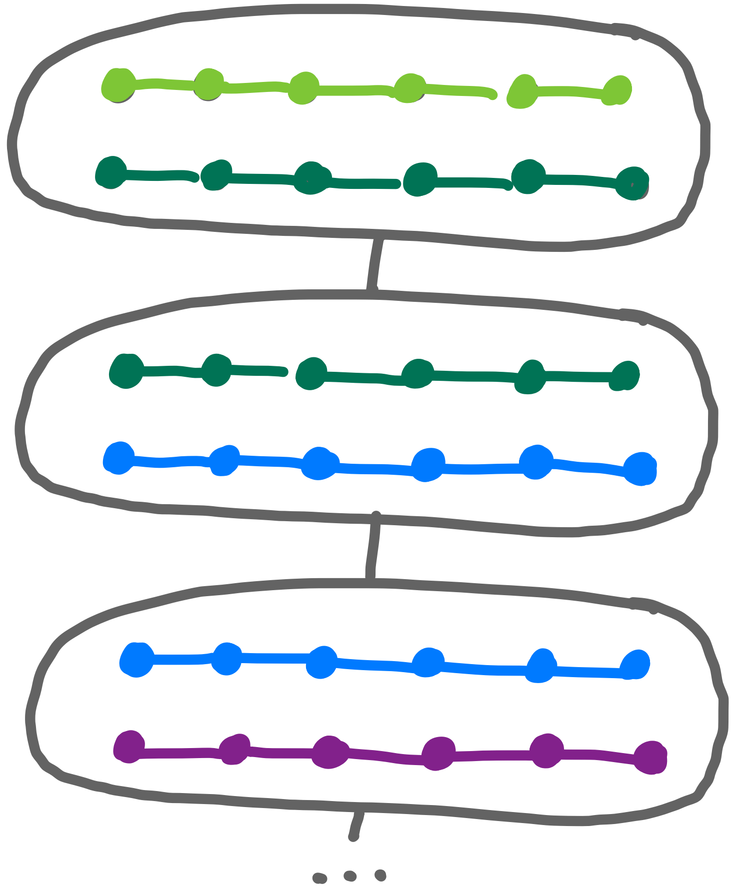


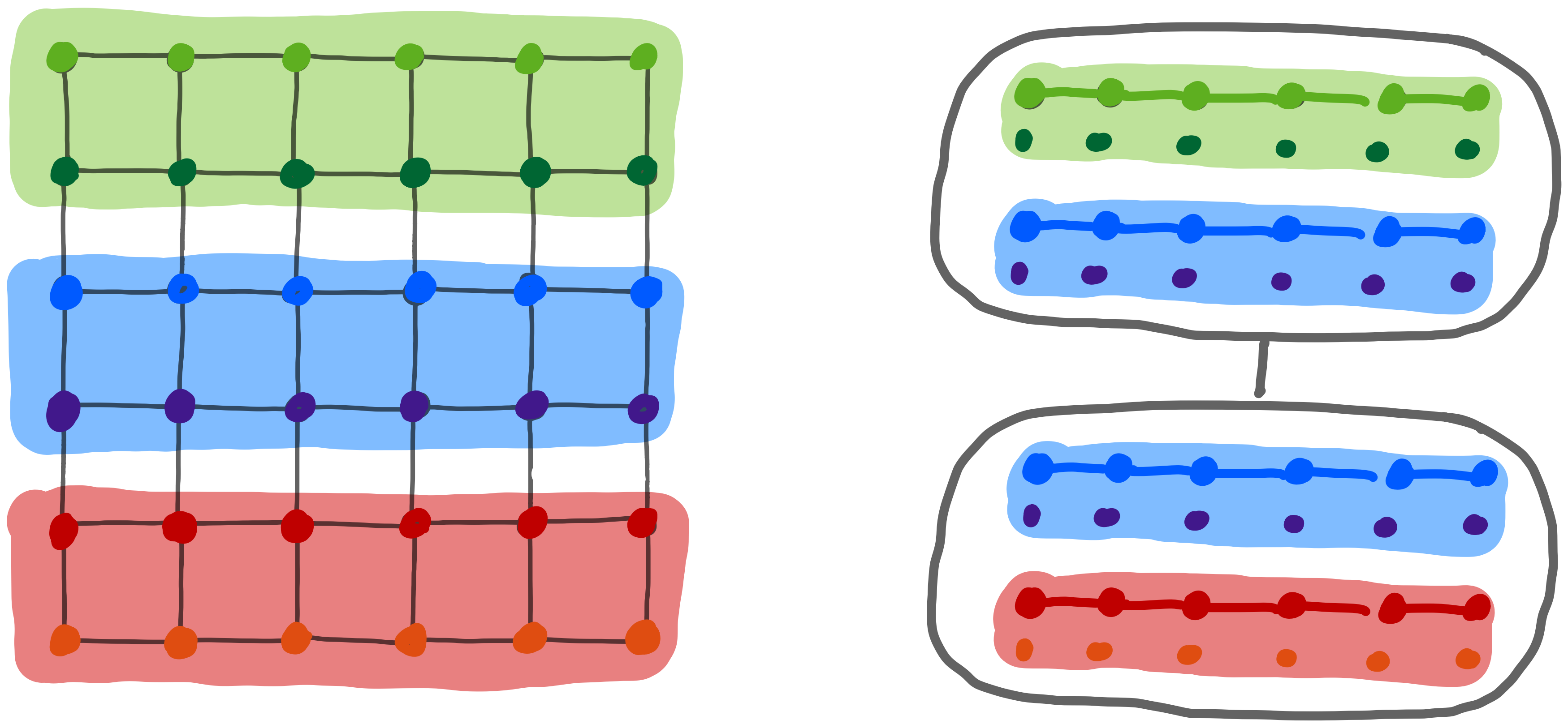


treewidth k: bags of $\leq k$ vertices $\quad O(\sqrt{n})$

cop-width k [And'98]: bags of $\leq k^2$ shortest* paths $\quad O(1)$

Cop-Decomposition

[Abraham-Gavoille-Gupta-Neiman-Talwar'14]

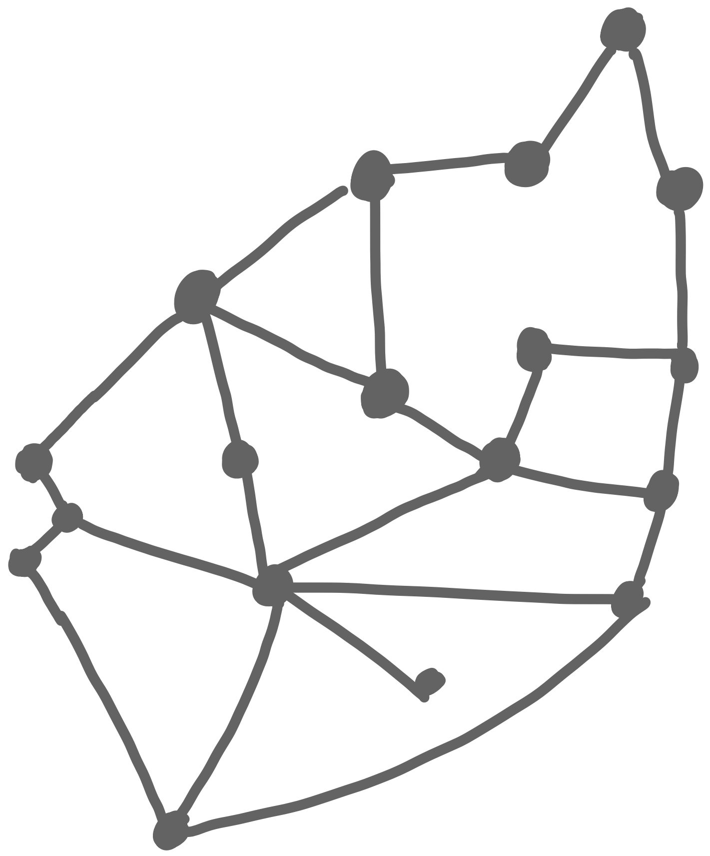


treewidth k: bags of $\leq k$ vertices $O(\sqrt{n})$

cop-width k [And'98]: bags of $\leq k^2$ shortest* paths $O(1)$

cop-width k [AGGNT'14]: bags of $\leq k^2$ expanded paths $O(1)$

Cop-Decomposition: Construction



Cop decomposition

- Select vertex v.
- $T \leftarrow$ SSSP tree from v to each visible cluster.
- Create cluster consisting of T and an $\varepsilon\Delta$ neighborhood.

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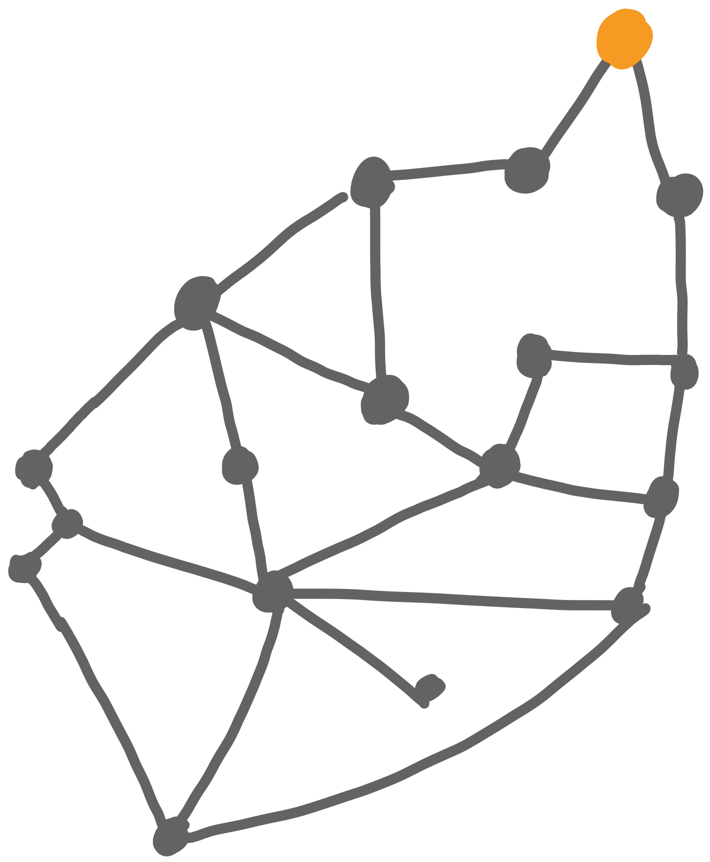

G

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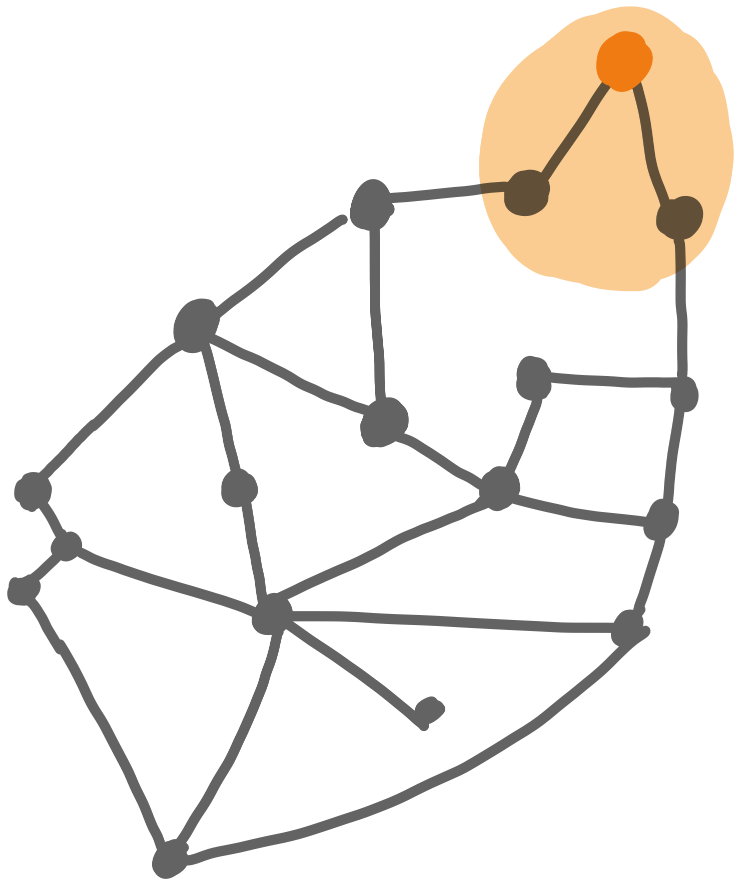

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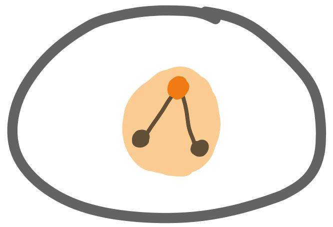

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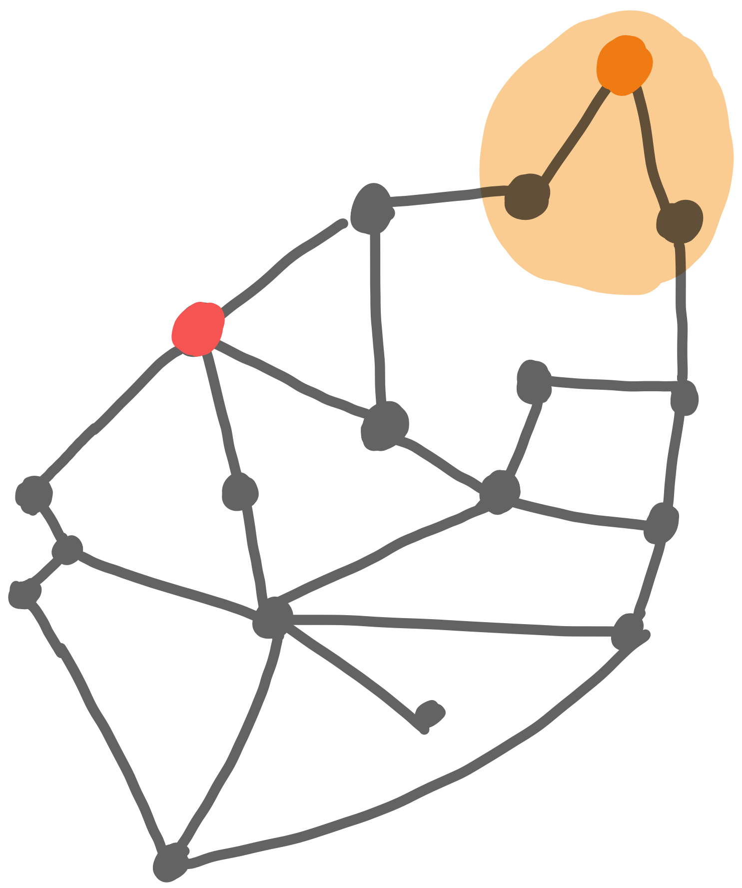

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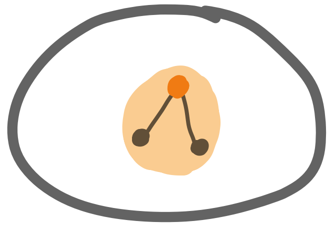

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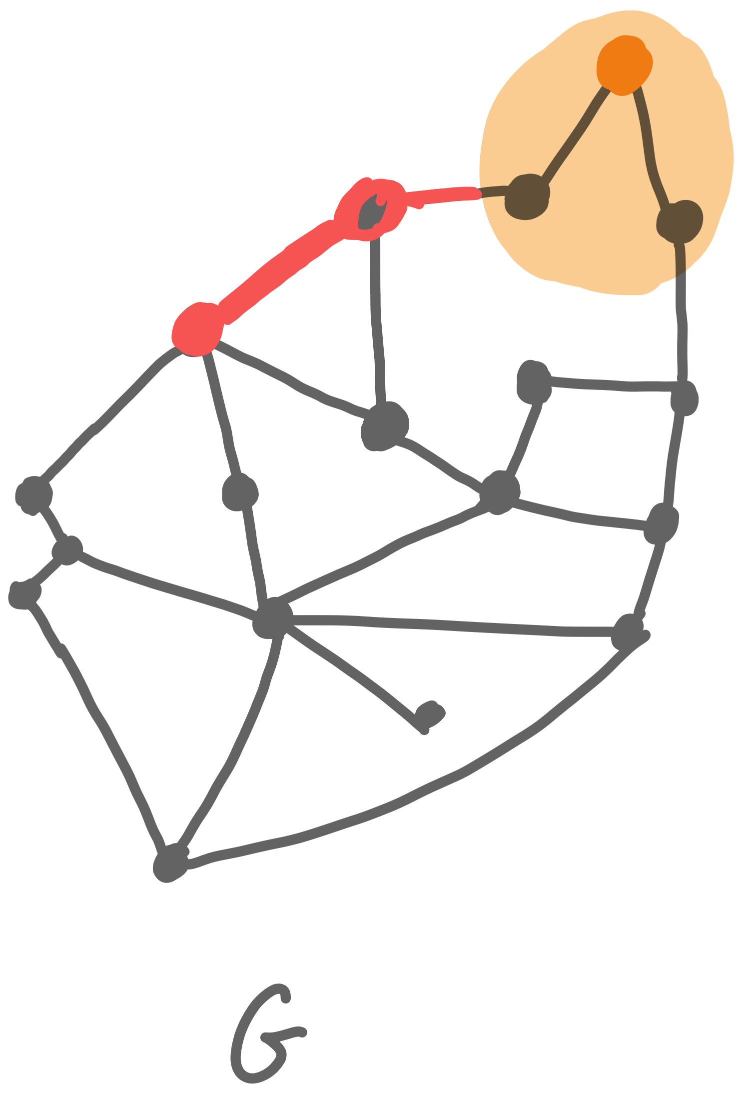

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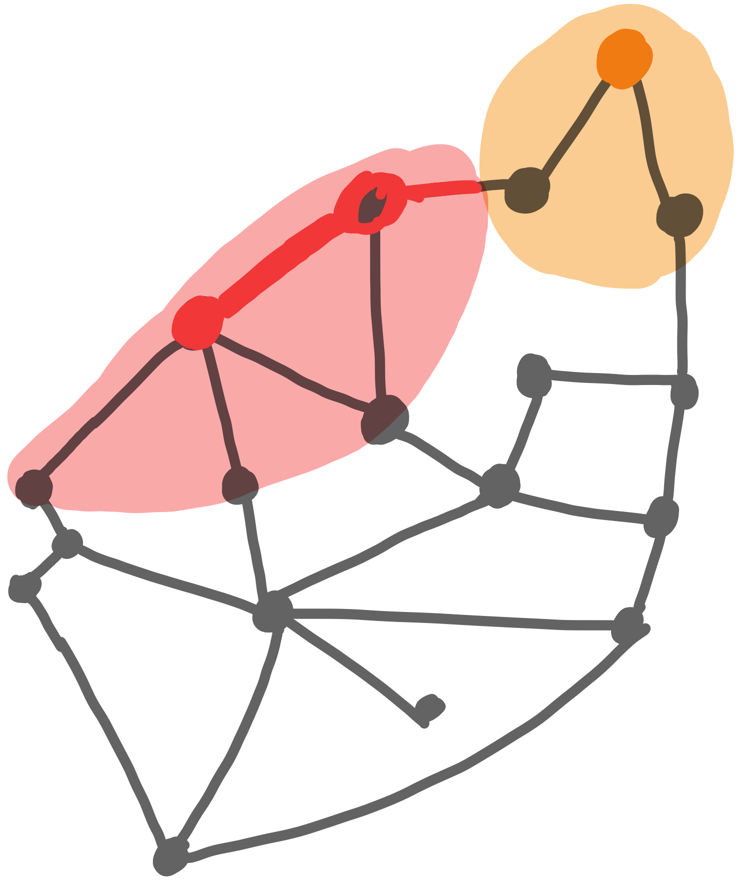

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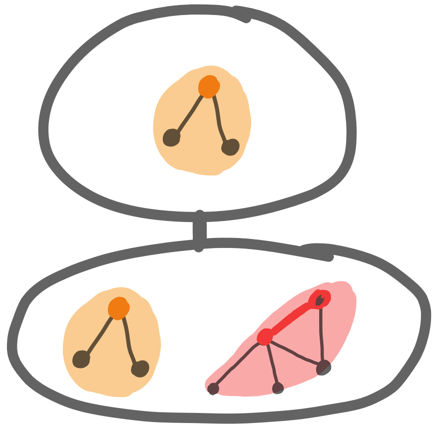

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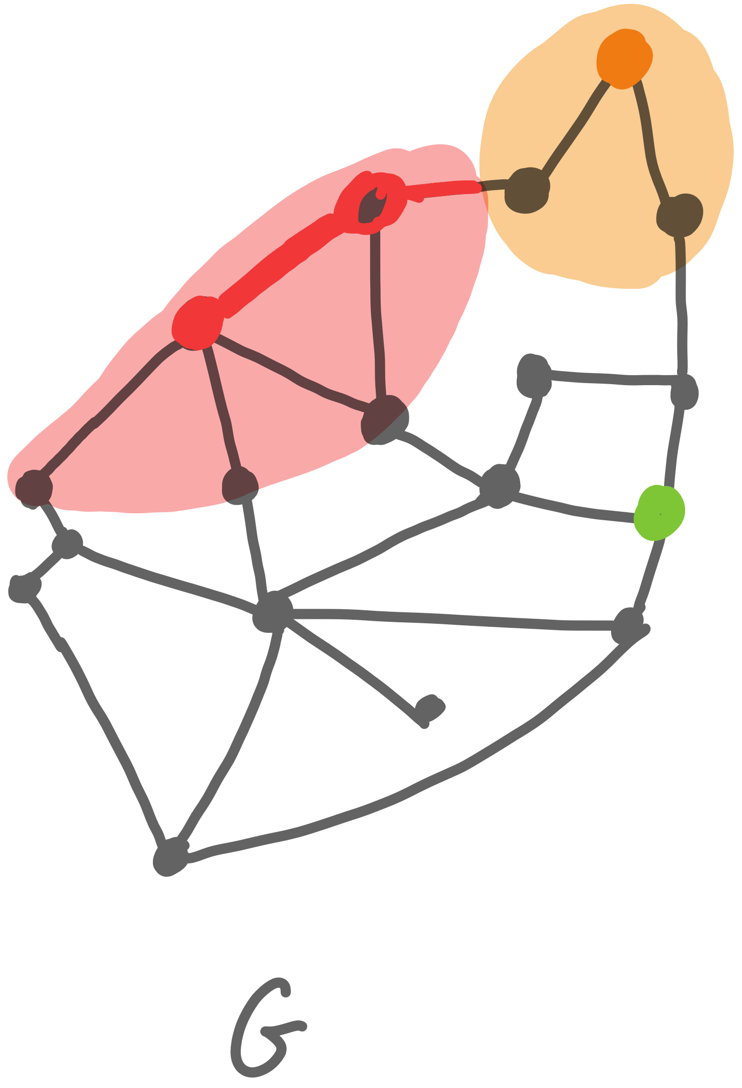

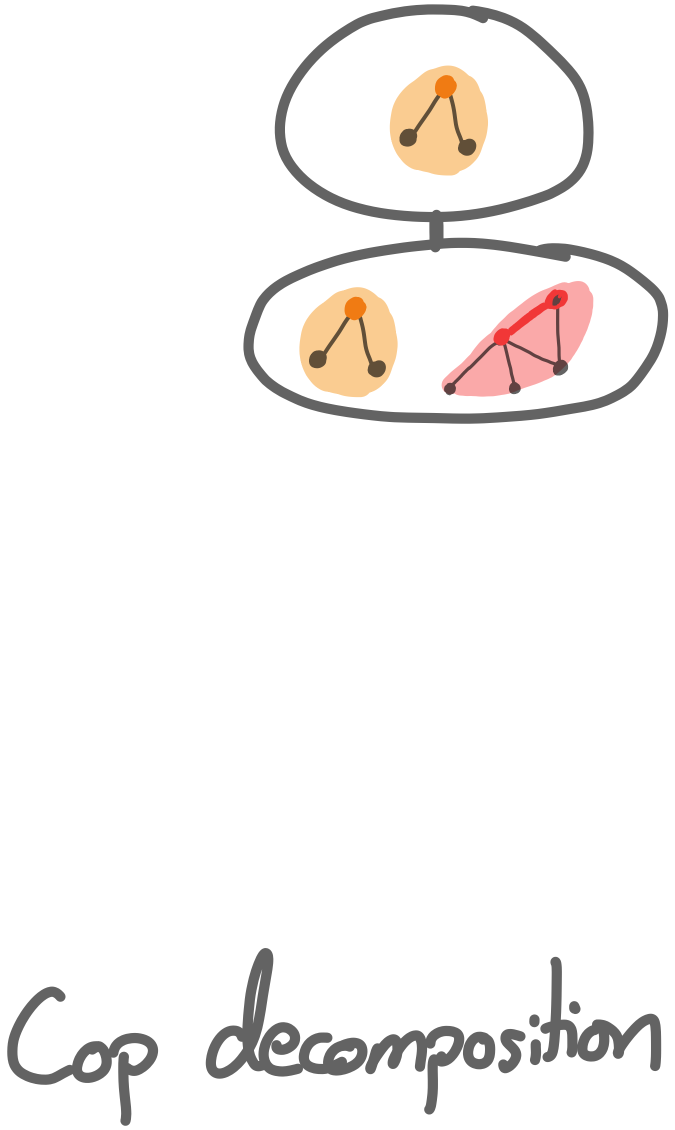

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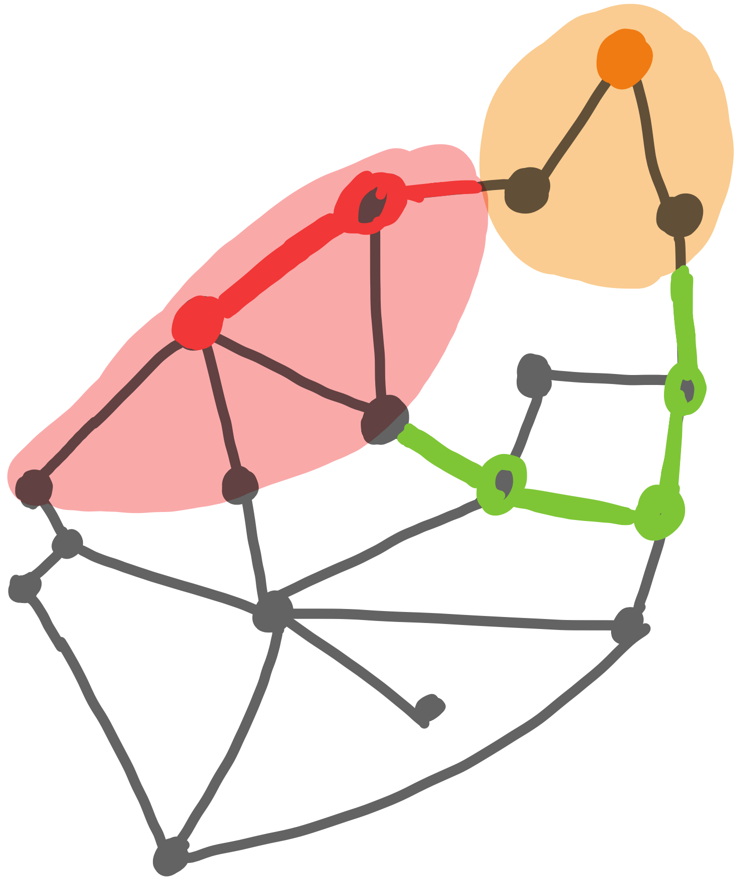

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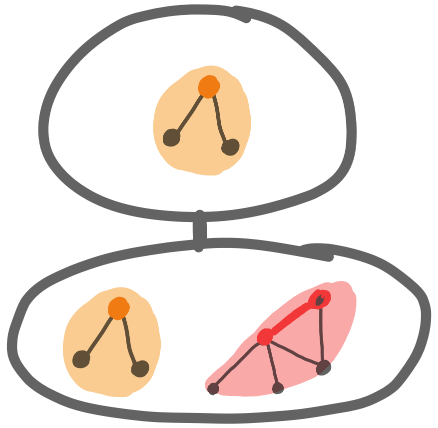

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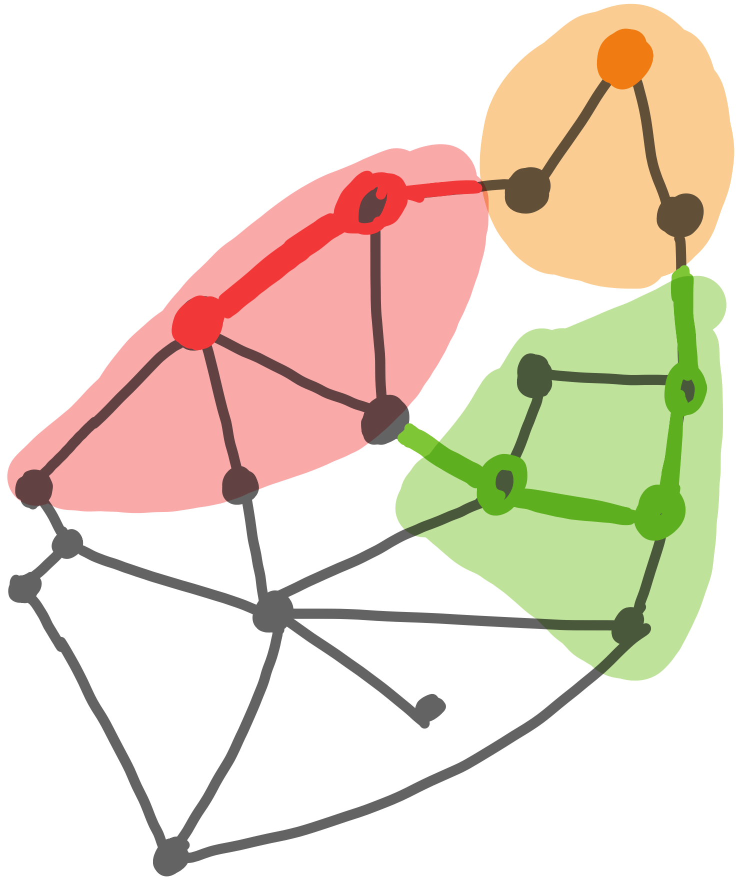

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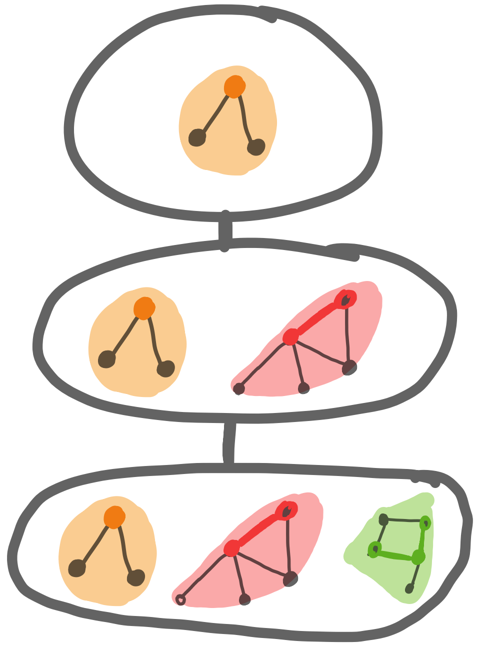

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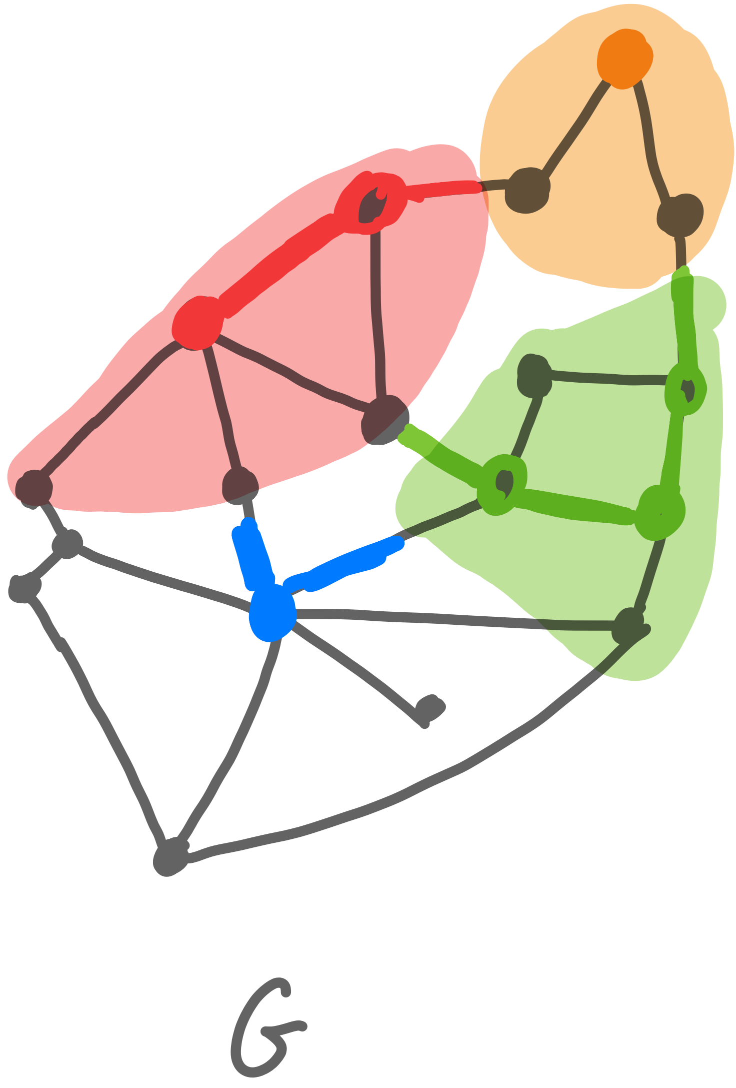

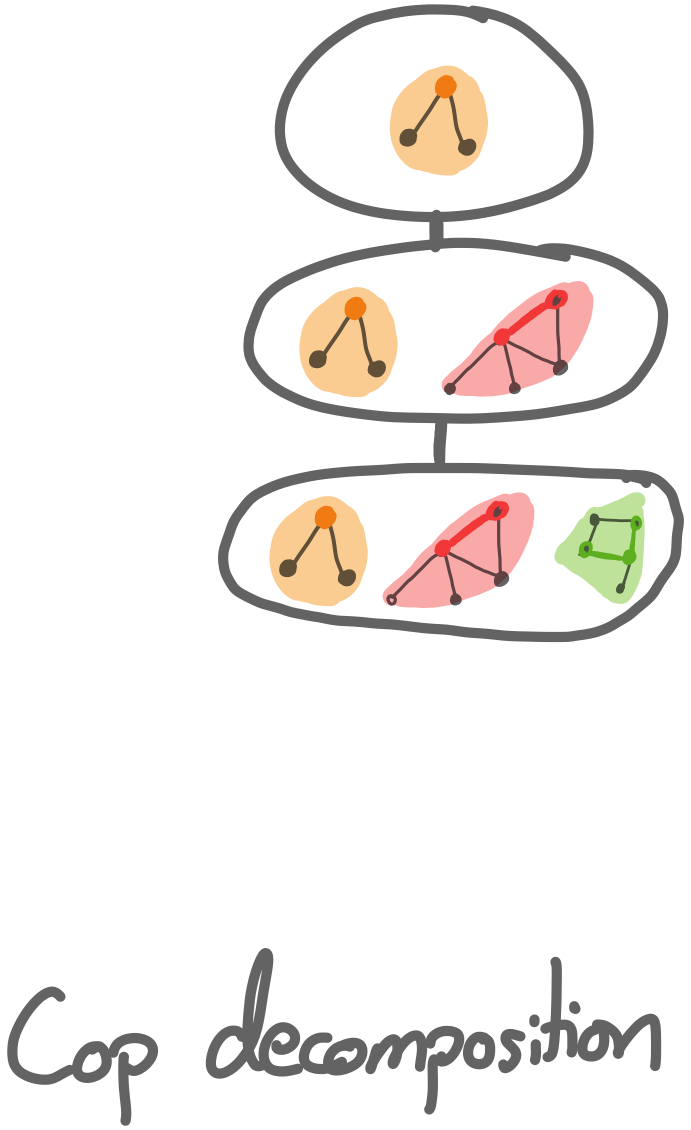

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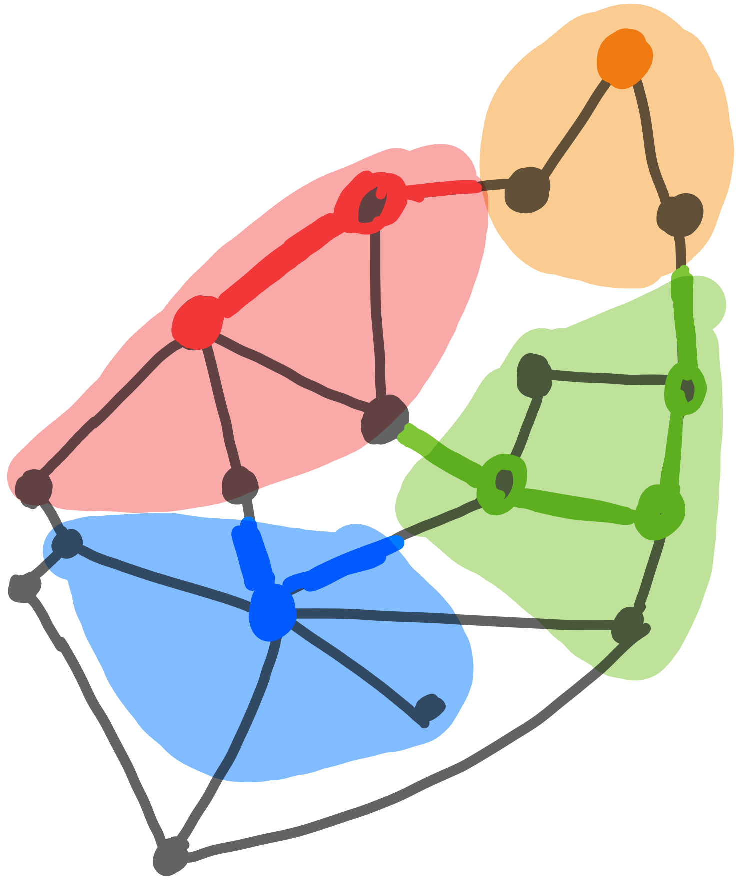

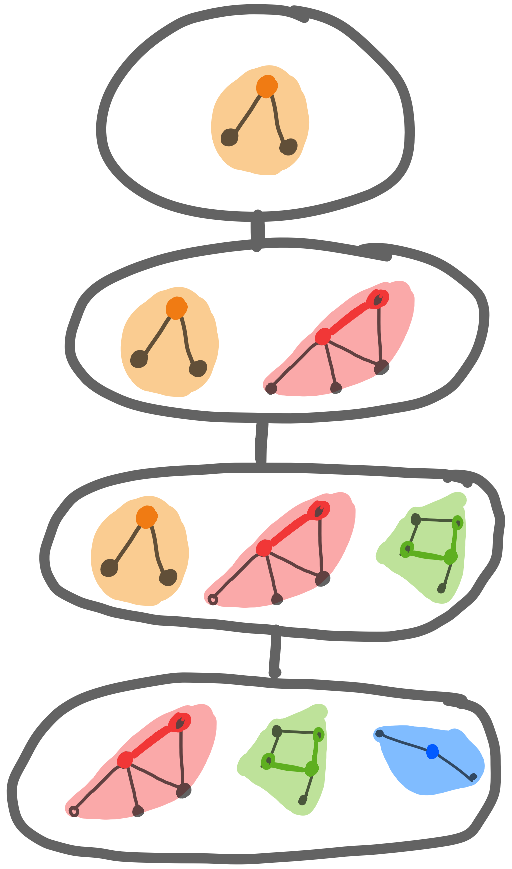

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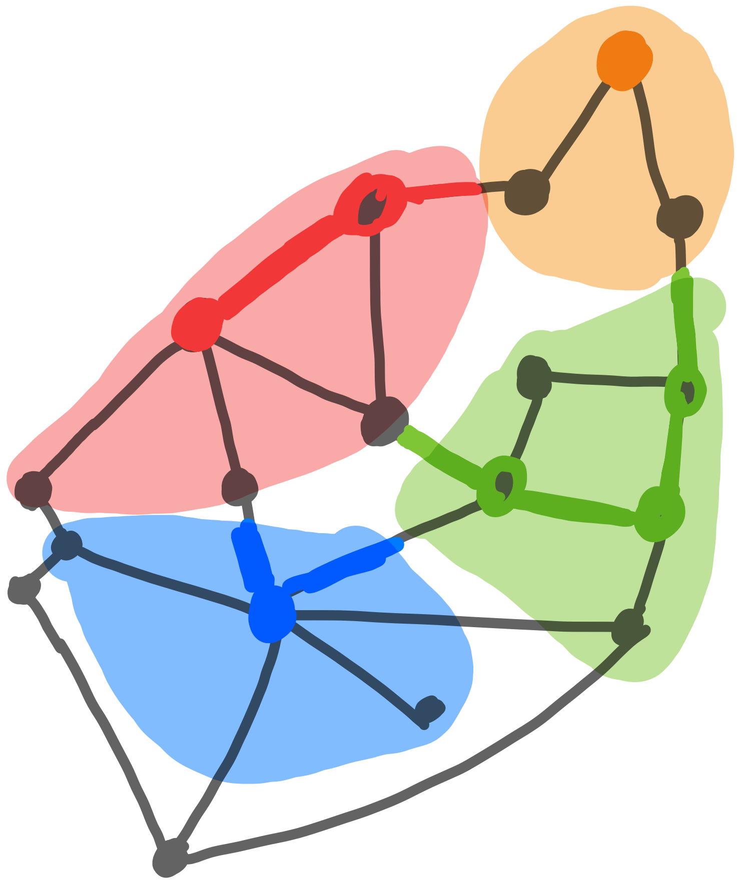
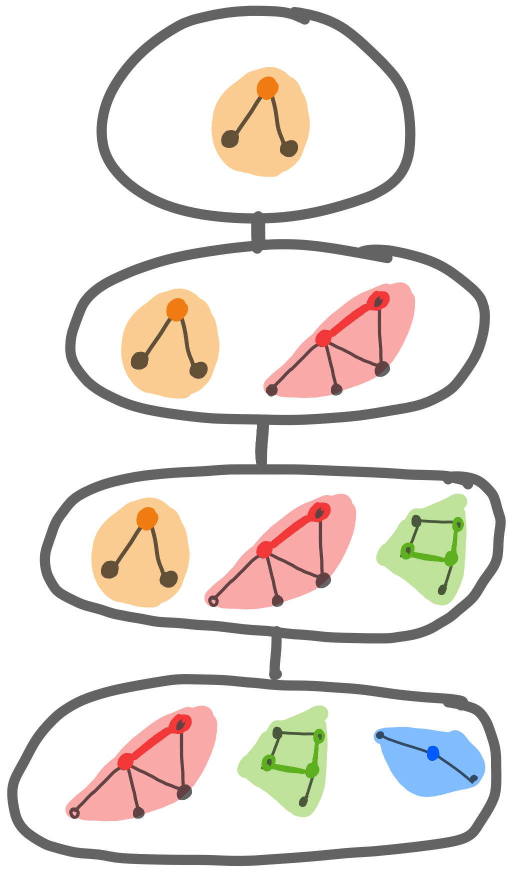
Cop-Decomposition: Buffer Property?





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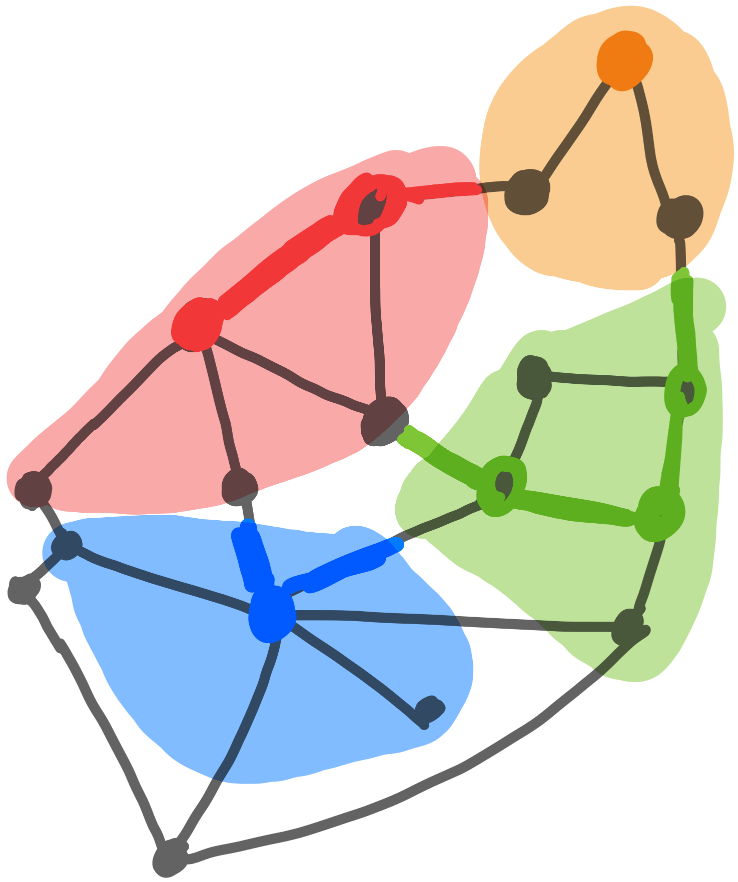

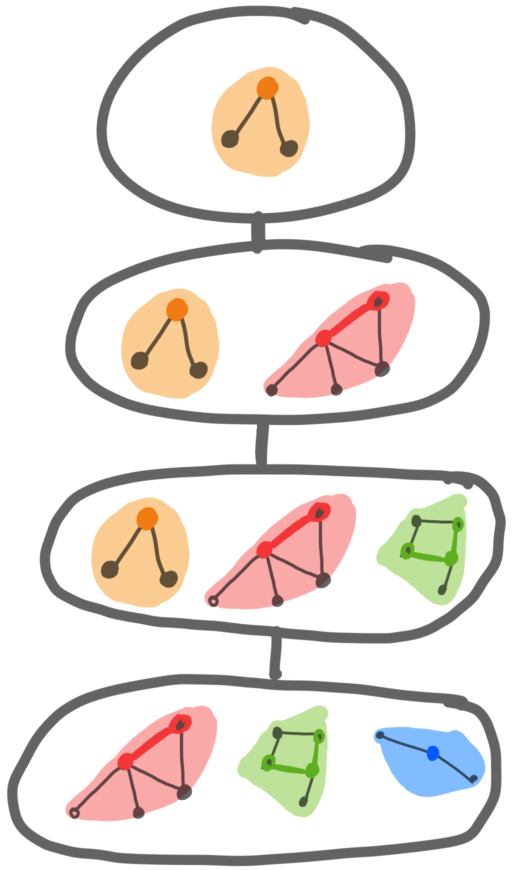

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- Create cluster consisting of T and an $O(\varepsilon\Delta)$ neighborhood.
  random

[AGGNT'14]: Buffer property holds *in expectation* with buffer $\varepsilon\Delta/r$, for K_r-minor-free graphs.

Outline

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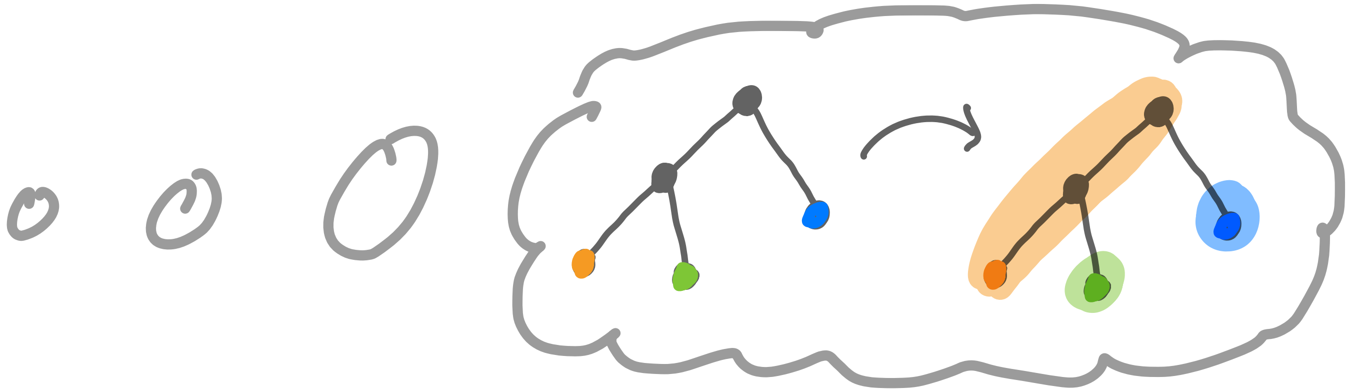

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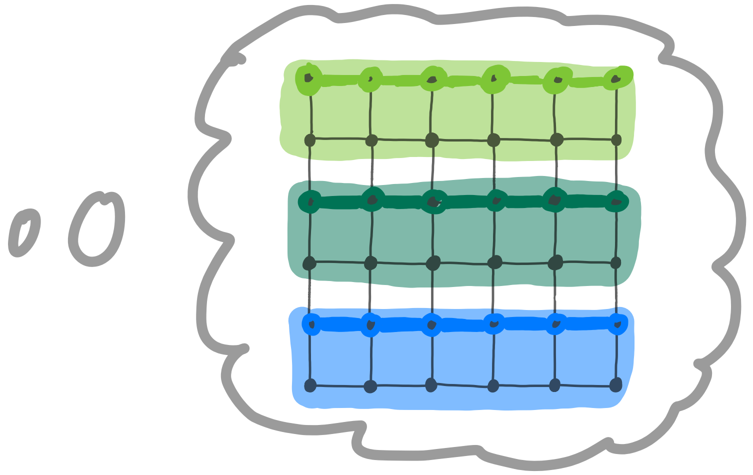
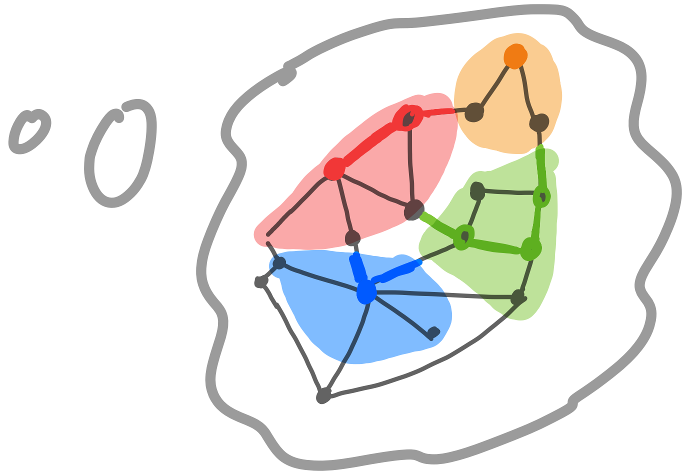
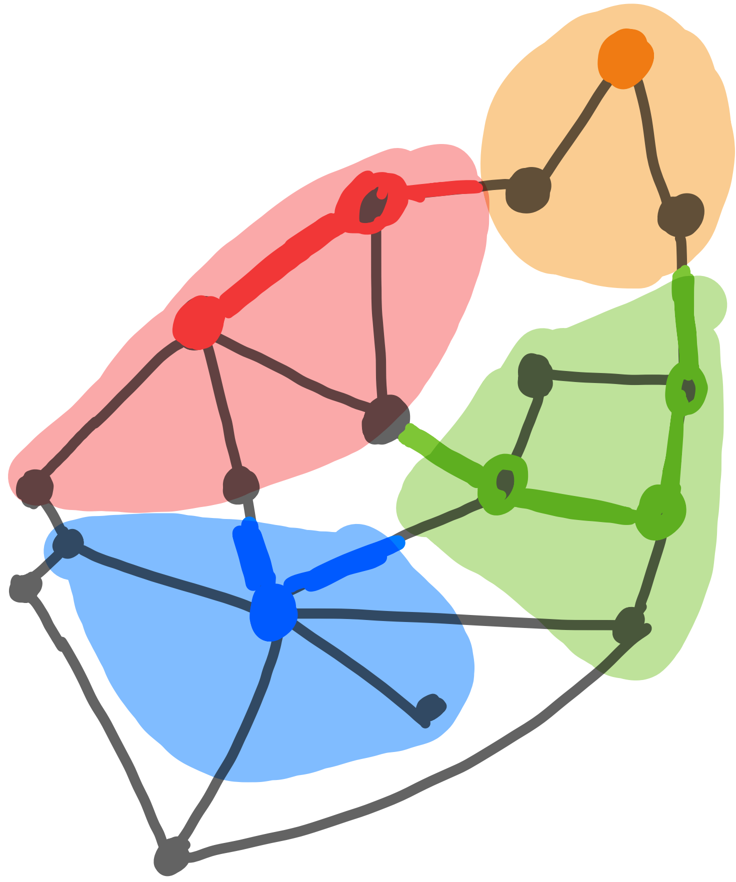


3. Cop-decomposition of [AGGNT'14]

4. Our deterministic modification

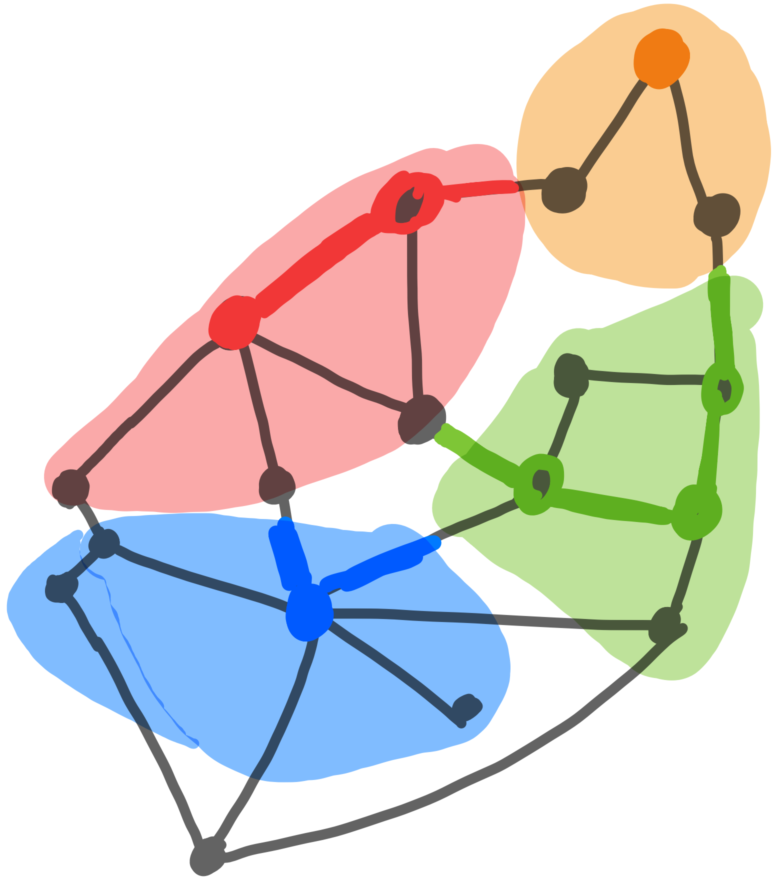


Buffered Cop-Decomposition: New Construction



- Select vertex v.
- $T \leftarrow$ SSSP tree from v to each visible cluster.
- Create cluster η consisting of T and an $\varepsilon\Delta/r$ neighborhood.
- If η cuts off old supernode, expand it by $\varepsilon\Delta/r$. Repeat.

Buffered Cop-Decomposition: New Construction



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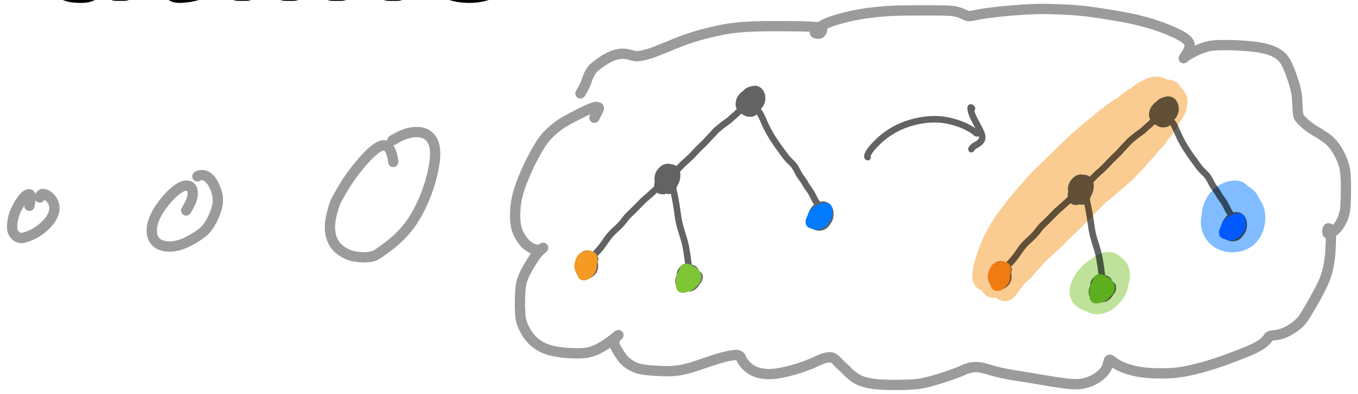
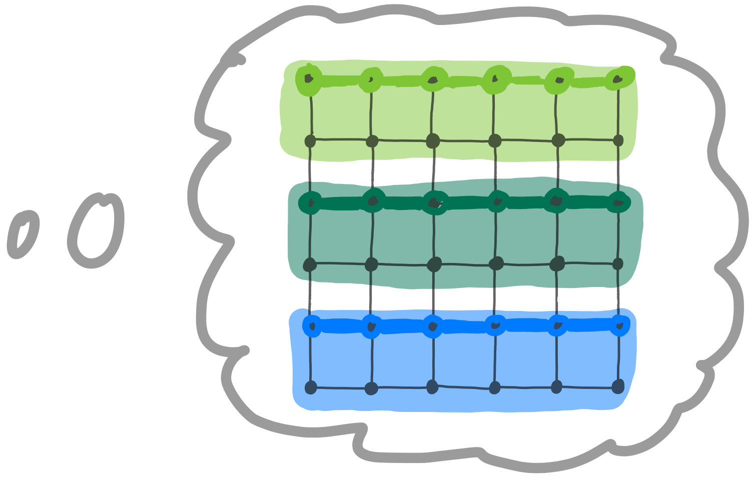
Outline

1. Steiner Point Removal



⇑

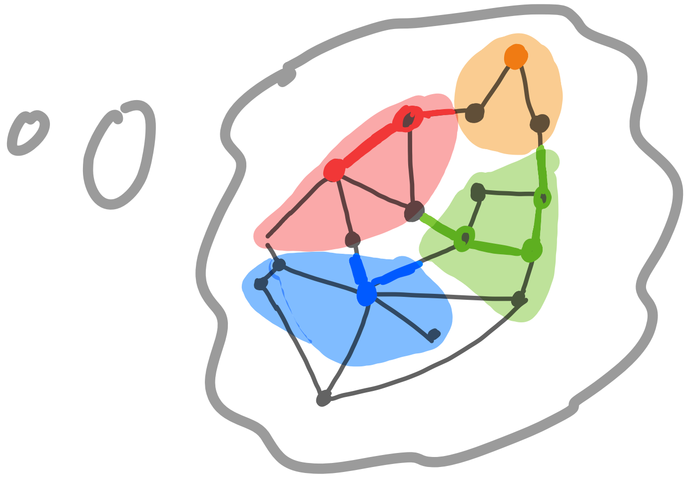
2. Shortcut partition & the buffer property



⇑

3. Cop-decomposition of [AGGNT'14]

4. Our deterministic modification



Conclusion

We show K_r-minor-free graphs admit $2^{O(r \log r)}$-distortion SPR solutions by constructing shortcut partition.

Open Questions

- Improved dependence on minor size? (i.e, $\operatorname{poly} r$ for K_r-minor-free or even treewidth-r graphs)
- What $O(1)$ distortion is achievable for planar graphs?
- Scattering (not shortcut) partition for planar/minor-free?

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Thank you!